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Generalized projection and approximation of fixed points of nonself maps

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Abstract

Let K be a nonempty closed convex *proper* subset of a real uniformly convex and uniformly smooth Banach space E; $T : K \to E$ be an *asymptotically weakly suppressive*, *asymptotically weakly contractive* or *asymptotically nonextensive* map with $F(T) := \{x \in K: Tx = x\} \neq \emptyset$. Using the notion of generalized projection, iterative methods for approximating fixed points of T are studied. Convergence theorems with estimates of convergence rates are proved. Furthermore, the stability of the methods with respect to perturbations of the operators and with respect to perturbations of the constraint sets are also established.

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1. Introduction

Let E be a real normed linear space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^*: \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

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where $\langle \cdot\,,\,\cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued and if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. We shall denote the single-valued duality mapping by j.

Let K be a subset of a Banach space E. A map $T: K \to K$ is called a *strict contraction* if there exists $k \in [0,1)$ such that $\|Tx - Ty\| \leqslant k\|x-y\|$, and is called *nonexpansive* if, for arbitrary $x,\ y \in K$, $\|Tx - Ty\| \leqslant \|x - y\|$. The map T is called *asymptotically nonexpansive* if, for each $x,\ y \in K$, we have $\|T^n x - T^n y\| \leqslant k_n\|x - y\|$, where $\{k_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} k_n = 1$. It is obvious that for asymptotically nonexpansive mappings it may be assumed that $k_n \geqslant 1$ and that $k_{i+1} \leqslant k_i,\ i = 1, 2, \ldots$.

Let K be a nonempty convex subset of a real normed linear space E. For strict contraction mappings, nonexpansive and asymptotically nonexpansive mappings T of K *into itself* with a fixed point in K, three well-known iterative methods, the celebrated *Picard method*, the *Mann iteration method* (see, for example, [15]) and the *Ishikawa iteration method* (see, for example, [13]), have successfully been employed to approximate such fixed points. If, however, the domain of T, $D(T)$, is a *proper* subset of E (and this is the case in several applications), and T *maps* K *into* E, these iteration methods may not be well defined. Under this situation, for Hilbert spaces, this problem has been overcome by the introduction of the *metric projection* in the recursion formulas (see, for example, [8–10]). The advantage of this is that if K is a nonempty closed convex subset of a Hilbert space H and $P_K: H \to K$ is the metric projection of H onto K, then P_K is *nonexpansive*. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_K in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Let E be a real normed linear space. Consider the functional defined by

$$V(x,\xi) := \|x\|^2 - 2\langle x, j(\xi)\rangle + \|\xi\|^2 \quad \text{for } x,\ \xi \in E, \tag{1.1}$$

where $j(\xi) \in J(\xi)$. Observe that, in a Hilbert space H, (1.1) reduces to $V(x,\xi) = \|x - \xi\|_H^2,\ x,\ \xi \in H$.

The *generalized projection* $\Pi_K: E \to K$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $V(x,\xi)$, that is, $\Pi_K x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$V(x,\bar{x}) := \inf_{\xi \in K} V(x,\xi). \tag{1.2}$$

Existence and uniqueness of the operator Π_K follow from the properties of the functional $V(x,\xi)$ and strict monotonicity of the mapping J (see, for example, [4]). In Hilbert space, $\Pi_K = P_K$.

Some properties of Π_K used in the sequel are the following (see, for example, [4]).

(a) The operator Π_K is the identity on K.

(b) The operator Π_K produces an absolutely best approximation of $x \in E$ relative to the functional $V(x, \xi)$, that is,

$$V(\Pi_K x, \xi) \leqslant V(x, \xi) - V(x, \Pi_K x), \quad \forall \xi \in K.$$

Consequently, Π_K is *the conditionally nonexpansive* operator relative to the functional $V(x, \xi)$ in Banach spaces, i.e. $V(\Pi_K x, \xi) \leqslant V(x, \xi)$, $\forall \xi \in K$.

(c) The operator Π_K is uniformly continuous on each bounded subset of E. Let $x,\ y \in E$, $||x|| \leqslant R$. $||y|| \leqslant R$. Then,

$$||\Pi_K x - \Pi_K y|| \leqslant 4LRg_E^{-1}(||J(x) - J(y)||/2R),$$

where g_E^{-1} is the inverse function to $g_E(\varepsilon)$ (defined in Section 2).

Let K be a nonempty subset of a Banach space E. A map $T : K \to E$ is called *strongly suppressive* on K if there exists $0 < q < 1$ such that for all $x,\ y \in K$,

$$V(Tx, Ty) \leqslant qV(x, y). \tag{1.3}$$

T is called *weakly suppressive of class* $C_{\psi(t)}$ if there exists a continuous and nondecreasing function $\psi(t)$ defined on $\mathbb{R}^+$ such that ψ is positive on $\mathbb{R}^+ \setminus \{0\}$, $\psi(0) = 0$, $\lim_{t \to \infty} \psi(t) = +\infty$ and $\forall x,\ y \in K$,

$$V(Tx, Ty) \leqslant V(x,\ y) - \psi(V(x, y)). \tag{1.4}$$

The map T is called *nonextensive* if

$$V(Tx, Ty) \leqslant V(x, y) \quad \forall x,\ y \in K. \tag{1.5}$$

It is trivial to see that in Hilbert spaces, nonextensive operators are nonexpansive, and vice versa; and strongly suppressive operators coincide with strict contractions. Alber and Guerre-Delabriere [4] (see also [2,3,5]) introduced the above classes of nonself maps and, assuming the existence of fixed points, proved convergence theorems with the help of the generalized projection maps.

It is our purpose in this paper to first introduce the classes of *asymptotically weakly suppressive*, *asymptotically nonextensive* and *asymptotically weakly contractive nonself maps* which are important generalizations of the classes of maps studied by Alber and Guerre-Delabriere [4]. Then, assuming the existence of fixed points for maps in our classes of operators, and using several results of Alber and Guerre-Delabriere [4], we prove convergence theorems with estimates of the convergence rates and establish the stability of our iterative methods with respect to perturbations of operators and with respect to perturbations of the constraint sets. Our theorems extend and improve several results of [4].

2. Preliminaries

A Banach space E is called *smooth* if for every $x \in E$ with $||x|| = 1$, there exists a unique $j \in E^*$ such that $||j|| = ||j(x)|| = 1$ (see, e.g., [14]). The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1,\ \ ||y|| = \tau\right\}.$$

A Banach space E is called *uniformly smooth* if and only if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$. It is known (see, e.g., [11,14,16]) that $\rho_E(\tau)$ is continuous, increasing and $\rho_E(0) = 0$. Moreover, $h_E(\tau) := \tau^{-1}\rho_E(\tau)$ is continuous, nondecreasing and $h_E(0) = 0$.

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right| : ||x|| = ||y|| = 1;\ \ \varepsilon = ||x-y||\right\}.$$

E is *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$.

It is well known (see, e.g., [11,14,16]) that $\delta_E(\varepsilon)$ is continuous, increasing and $\delta_E(0) = 0$. Moreover, $g_E(\varepsilon) := \varepsilon^{-1}\delta_E(\varepsilon)$ is continuous and increasing.

Let K_1 and K_2 be convex bounded and closed sets. It is known that if $\mathscr{H}(K_1, K_2) \leqslant \sigma$, where

$$\mathscr{H}(K_1, K_2) := \max\left\{\sup_{z_1\in K_1}\ \inf_{z_2\in K_2}\ ||z_1 - z_2||,\ \ \sup_{z_1\in K_2}\ \inf_{z_2\in K_1}\ ||z_1 - z_2||\right\}$$

is the Hausdorff distance between K_1 and K_2, then the following lemma is valid.

Lemma 2.1 (Alber [1], Alber and Notik [6]). *If E is a uniformly convex space, $\delta_E(\varepsilon)$ its modulus of convexity, and $\delta_E^{-1}(\cdot)$ is the inverse function, then*

$$||P_{K_1}x - P_{K_2}x|| \leqslant C_1\delta_E^{-1}(4L(d+r)\sigma), \tag{2.1}$$

where P_K is a metric projection on the set E, $r = ||x||$, $d = \max\{d_1, d_2\}$, $d_i =$ dist(θ, K_i), $i = 1, 2$, θ is the origin of E, $C_1 = 2\max\{1, r+d\}$.

In the sequel, we shall also make use of the following lemmas.

Lemma 2.2 (Alber [1]). *If E is a uniformly convex and uniformly smooth Banach space, then the inequalities*

$$8C^2\delta_E(||x-y||/4C) \leqslant V(x,y) \leqslant 4C^2\rho_E(4||x-y||/C) \tag{2.2}$$

hold for all x and y in E, where $C = \sqrt{(||x||^2 + ||y||^2)/2}$. If $||x|| \leqslant R$ and $||y|| \leqslant R$, then

$$2L^{-1}R^2\delta_E(||x-y||/4R) \leqslant V(x,y) \leqslant 4LR^2\rho_E(4||x-y||/R). \tag{2.3}$$

Lemma 2.3 (Alber [1], Alber and Notik [6]). *If E is a uniformly convex and uniformly smooth Banach space and if $||x|| \leqslant R$ and $||y|| \leqslant R$, then*

$$(2L)^{-1}R^2\delta_E(||x-y||/2R) \leqslant \langle Jx - Jy, x-y\rangle \leqslant 2LR^2\rho_E(4||x-y||/R) \tag{2.4}$$

and

$$||Jx - Jy||_{E^*} \leqslant 8Rh_E(16L||x-y||/R), \tag{2.5}$$

where $1 < L < 1.7$ *hold.*

Lemma 2.4 (Alber and Guerre-Delabriere [3], Alber and Reich [7]). *Let* $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ *and* $\{\gamma_n\}$ *be sequences of nonnegative numbers such that* $\{\alpha_n\} \subseteq (0,1]$, $\sum \alpha_n = \infty$, $\sum \beta_n < \infty$ *and* $\frac{\gamma_n}{\alpha_n} \to 0$ *as* $n \to \infty$.

If the recursive inequality

$$\lambda_{n+1} \leqslant (1+\beta_n)\lambda_n - \alpha_n \psi(\lambda_n) + \gamma_n, \quad n = 1, 2, \ldots$$

is given, where $\psi(\lambda)$ *is a continuous and nondecreasing function* $\psi(t)$ *defined on* $\mathbb{R}^+$ *such that* ψ *is positive on* $\mathbb{R}^+ \setminus \{0\}$, $\psi(0) = 0$, $\lim_{t\to\infty} \psi(t) = +\infty$. *Then,*

(1) $\lambda_n \to 0$ *as* $n \to \infty$,
(2) *there exists a subsequence* $\{\lambda_{n_l}\} \subset \{\lambda_n\}$, $l \geqslant 1$, *such that*

$$\lambda_{n_l} \leqslant \psi^{-1}\left(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\bar{\gamma}_{n_l}}{\alpha_{n_l}},\right), \quad \text{where } \bar{\gamma}_{n_l} = \gamma_{n_l} + \beta_{n_l} M, \quad \text{for some } M > 0, \tag{2.16}$$

$$\lambda_{n_l+1} \leqslant \psi^{-1}\left(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\bar{\gamma}_{n_l}}{\alpha_{n_l}}\right) + \bar{\gamma}_{n_l}, \tag{2.17}$$

$$\lambda_n \leqslant \lambda_{n_l+1} - \sum_{n_l+1}^{n-1} \frac{\alpha_m}{\mathscr{A}_m}, \quad n_l + 1 \leqslant n < n_{l+1}, \quad \mathscr{A}_m = \sum_1^{m-1} \alpha_i, \tag{2.18}$$

$$\lambda_{n+1} \leqslant \lambda_1 - \sum_1^n \frac{\alpha_m}{\mathscr{A}_m} \leqslant \lambda_1, \quad 1 \leqslant n \leqslant n_1 - 1, \tag{2.19}$$

$$1 \leqslant n_1 \leqslant s_{\max} = \max\left\{ s : \sum_1^s \frac{\alpha_m}{\mathscr{A}_m} \leqslant \lambda_1 \right\}. \tag{2.20}$$

3. Main results

3.1. Successive approximations in Banach spaces

In this section, we give new definitions and prove our main theorems.

Definition 3.1. Let K be a nonempty subset of a real Banach space E. A map $T : K \to E$ is called *asymptotically weakly suppressive of class* $C_{\psi(t)}$ if there exists a continuous and nondecreasing function $\psi(t)$ defined on $\mathbb{R}^+$ such that ψ is positive on $\mathbb{R}^+ \setminus \{0\}$, $\psi(0) = 0$, $\lim_{t\to\infty} \psi(t) = +\infty$ and $\forall x, y \in K$ there exists $\{k_n\} \subseteq [1, \infty)$

with $\lim_{n\to\infty} k_n = 1$, such that

$$V(T(\Pi_K T)^{n-1}x,\ T(\Pi_K T)^{n-1}y) \leqslant k_n V(x,y) - \psi(V(x,y)), \quad \forall n \geqslant 1. \tag{3.1}$$

Let $F(T) : \{x \in K : Tx = x\}$, then T is called *asymptotically weakly hemi-suppressive* if $F(T) \neq \emptyset$ and inequality (3.1) holds for every $x \in K$ and $y \in F(T)$.

The map $T : K \to E$ is called *asymptotically nonextensive* if, for all $x,\ y \in K$, there exists $k_n \geqslant 1$, with $\lim_{n\to\infty} k_n = 1$, such that

$$V(T(\Pi_K T)^{n-1}x,\ T(\Pi_K T)^{n-1}y) \leqslant k_n V(x,y), \quad \forall\ n \geqslant 1, \tag{3.2}$$

and it is *called asymptotically quasi-nonextensive*, *if* $F(T) \neq \emptyset$ *and inequality* (3.2) *holds for every* $x \in K$ *and* $y \in F(T)$.

Remark 3.2. It is easy to see that the class of weakly suppressive maps with fixed points is a subclass of the class of asymptotically weakly hemi-suppressive maps; and the class of nonextensive maps with fixed points is a subclass of class of asymptotically quasi-nonextensive maps. Furthermore, we observe that, in Hilbert spaces and for self-maps, our definition of asymptotically nonextensive maps coincides with the definition of asymptotically nonexpansive maps introduced by Göebel and Kirk [12] and studied by various authors.

We now prove the following theorems.

Theorem 3.3. *Let* K *be a closed convex subset of a uniformly convex and uniformly smooth Banach space* E. *Let* $T : K \to E$ *be an asymptotically weakly suppressive operator of class* $C_{\psi(t)}$ *with sequence* $\{k_n\} \subseteq [1,\ \infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Suppose* $F(T) \neq \emptyset$ *and for arbitrary* $x_1 \in K$ *let the sequence* $\{x_n\}$ *be defined by*

$$x_{n+1} := (\Pi_K T)^n x_n, \quad n \geqslant 1. \tag{3.3}$$

Then, $\{x_n\}$ *converges strongly to some* $x^* \in F(T)$.

Proof. Let $x^* \in F(T)$. Set $\beta_n := k_n - 1$. Then, by the definition of asymptotically weakly suppressive map and property of Π_K, we have that

$$\begin{aligned} V(x_{n+1}, x^*) &= V((\Pi_K T)^n x_n, (\Pi_K T)^n x^*) \\ &\leqslant V(T(\Pi_K T)^{n-1}x_n, T(\Pi_K T)^{n-1}x^*) \\ &\leqslant k_n V(x_n, x^*) - \psi(V(x_n, x^*)) \\ &= (1+\beta_n)V(x_n, x^*) - \psi(V(x_n, x^*)) \leqslant \exp\left(\sum_{j=1}^{n} \beta_j\right) V(x_1, x^*), \end{aligned} \tag{3.4}$$

so that $V(x_n, x^*)$ is bounded. If we now set $\lambda_n := V(x_n, x^*)$, Lemma 2.4 and inequality (3.4) yield $\lim_{n\to\infty} V(x_n, x^*) = 0$. Consequently, from Lemma 2.2, there exists $R>0$ such that

$$\lim_{n\to\infty} \delta_E(||x_n - x^*||/4R) = 0.$$

Then, by the property of $\delta_E(\xi)$, $\lim_{n\to\infty} ||x_n - x^*|| = 0$. The proof is complete. □

It follows from the above proof that Theorem 3.3 is valid for asymptotically weakly hemi-suppressive maps. Thus, from Remark 3.2 we have that Theorem 2.7 of [4] is a special case of Theorem 3.3 in which $k_n = 1$ for all positive integers n.

Theorem 3.4. *Let K be a closed convex subset of a uniformly smooth and uniformly convex Banach space E. Let $T : K \to E$ be an asymptotically nonextensive operator with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose $F(T) \neq \emptyset$ and for arbitrary $x_1 \in K$ let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} := (\Pi_K T)^n x_n, \quad n \geqslant 1. \tag{3.5}$$

(i) *If the operator $A := I - T$ is demi-closed and $||x_{n+1} - x_n|| \to 0$, then $\lim_{n\to\infty} Ax_n = 0$ and all weak accumulation points of $\{x_n\}$ belong to the fixed point set $F(T)$ of T.*

(ii) *In addition, if either $F(T)$ is a singleton, or the duality mapping j is weakly sequentially continuous (on some bounded set containing $\{x_n\}$), then $\{x_n\}$ converges weakly to a point $x^* \in F(T)$.*

Proof. Let $x^* \in F(T)$ and set $\beta_n := k_n - 1$. Then, from (3.5) and property of Π_K, we get that

$$\begin{aligned} V(x_{n+1}, x^*) &= V((\Pi_K T)^n x_n, (\Pi_K T)^n x^*) \\ &\leqslant V(T(\Pi_K T)^{n-1} x_n, x^*) - V(T(\Pi_K T)^{n-1} x_n, (\Pi_K T)^n x_n) \\ &\leqslant k_n V(x_n, x^*) - V(T(\Pi_K T)^{n-1} x_n, (\Pi_K T)^n x_n) \\ &= (1 + \beta_n) V(x_n, x^*) - V(T(\Pi_K T)^{n-1} x_n, (\Pi_K T)^n x_n) \\ &\leqslant \exp\left(\sum_{j=1}^{n} \beta_j\right) V(x_1, x^*). \end{aligned} \tag{3.6}$$

This implies that $V(x_n, x^*)$ is bounded. From (3.6), we have

$$V(T(\Pi_K T)^{n-1} x_n, (\Pi_K T)^n x_n) \leqslant (1 + \beta_n) V(x_n, x^*) - V(x_{n+1}, x^*)$$

and hence $\sum V(T(\Pi_K T)^{n-1} x_n, (\Pi_K T)^n x_n) < \infty$. Thus, as in the proof of Theorem 3.3, we get that

$$\lim_{n\to\infty} ||x_{n+1} - T(\Pi_K T)^{n-1} x_n|| = 0. \tag{3.7}$$

This, with our assumption, gives

$$||T(\Pi_K T)^{n-1} x_n - x_n|| \leqslant ||T(\Pi_K T)^{n-1} x_n - x_{n+1}|| + ||x_n - x_{n+1}|| \to 0. \tag{3.8}$$

Now, we show that $||Tx_n - x_n|| \to 0$ as $n \to \infty$. But

$$\begin{aligned}||Tx_n - x_n|| &= ||Tx_n - T(\Pi_K T)^{n-1}x_n + T(\Pi_K T)^{n-1}x_n - x_n|| \\ &\leqslant ||Tx_n - T(\Pi_K T)^{n-1}x_n|| + ||T(\Pi_K T)^{n-1}x_n - x_n|| \\ &= ||T(\Pi_K T)^{n-1}x_{n-1} - T(\Pi_K T)^{n-1}x_n|| + ||T(\Pi_K T)^{n-1}x_n - x_n||. \quad (3.9)\end{aligned}$$

Since by assumption $||x_{n+1} - x_n|| \to 0$, we have by Lemma 2.2 that

$$V(x_{n+1}, x_n) \leqslant 4C^2 \rho_E(4||x_{n+1} - x_n||/C) \to 0, \quad \text{for some } C > 0,$$

which implies $V(x_{n+1}, x_n) \to 0$ as $n \to \infty$; and hence, $V(x_{n-1}, x_n) \to 0$ as $n \to \infty$.

Again by Lemma 2.2, and the asymptotic nonextensive property of T, we have

$$\begin{aligned}&8C^2\delta_E(||T(\Pi_K T)^{n-1}x_{n-1} - T(\Pi_K T)^{n-1}x_n||/4C) \\ &\quad \leqslant V(T(\Pi_K T)^{n-1}x_{n-1}, T(\Pi_K T)^{n-1}x_n) \\ &\quad \leqslant k_n V(x_{n-1}, x_n) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} ||T(\Pi_K T)^{n-1}x_{n-1} - T(\Pi_K T)^{n-1}x_n|| = 0. \tag{3.10}$$

Using (3.8) and (3.10), inequality (3.9) gives $||Tx_n - x_n|| \to 0$ as $n \to \infty$. The remainder of the proof follows as in the proof of Theorem 2.8 of [4]. □

In what follows, we study the iterative method with perturbed maps $T_n : K \to E$ defined by (3.11)

$$y_{n+1} = (\Pi_K T_n)^n y_n, \quad n \geqslant 1. \tag{3.11}$$

Before we state and prove our next theorem, we first introduce the following definition.

Definition 3.5. Let K be a nonempty subset of a real Banach space E. A map $T : K \to E$ is called *asymptotically weakly contractive of class* $C_{\psi(t)}$ if there exists a real sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and there exists $\psi(t)$ as in Definition 3.1 such that

$$||T(\Pi_K T)^{n-1}x - T(\Pi_K T)^{n-1}y|| \leqslant k_n||x - y|| - \psi(||x - y||), \quad \forall x, y \in K.$$

Theorem 3.6. *Let K be a closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : K \to E$ be a map such that $T(\Pi_K T)^{n-1}$ is bounded and $\Pi_K T : K \to K$ is asymptotically weakly contractive of class $C_{\psi(t)}$ with $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and let $x^* \in F(T)$. Suppose that there exist sequences of positive numbers $\{\delta_n\}$ and $\{h_n\}$ converging to zero as $n \to \infty$, a finite positive function $g(t)$ defined on $\mathbb{R}^+$ such that for all $n \geqslant 1$,*

$$||T_n(\Pi_K T_n)^{n-1}v - T(\Pi_K T)^{n-1}v|| \leqslant h_n g(||v||) + \delta_n, \quad \forall v \in K. \tag{3.12}$$

(1) *If the iterative sequence* (3.11), *starting at an arbitrary* $y_1 \in K$ *is bounded*, *say by* $M>0$, *or*

(2) *if* $\lim_{n\to\infty} \sigma_n = 0$, *where* $\sigma_n = ||(\Pi_K T_n)^n y_n - (\Pi_K T)^n y_n||$, *then it converges in norm to the point* x^*. *Moreover*, *there exists a subsequence* $\{y_{n_l}\} \subset \{y_n\}$, $l \geqslant 1$, *such that*

$$||y_{n_l} - x^*|| \leqslant \psi^{-1}\left(\frac{1}{n_l} + \bar{\gamma}_{n_l}\right), \tag{3.13}$$

where $\bar{\gamma}_n$ *is given by* (3.19). *Furthermore*,

$$||y_{n_l+1} - x^*|| \leqslant \psi^{-1}\left(\frac{1}{n_l} + \bar{\gamma}_{n_l}\right) + \bar{\gamma}_{n_l}, \tag{3.14}$$

$$||y_n - x^*|| \leqslant ||y_{n_l+1} - x^*|| - \sum_{n_l+1}^{n-1} \frac{1}{m}, \quad n_l + 1 \leqslant n < n_{l+1}, \tag{3.15}$$

$$||y_{n+1} - x^*|| \leqslant ||y_1 - x^*|| - \sum_{1}^{n} \frac{1}{m} \leqslant ||y_1 - x^*||, \quad 1 \leqslant n \leqslant n_1 - 1, \tag{3.16}$$

$$1 \leqslant n_1 \leqslant s_{\max} = \max\left\{ s : \sum_{1}^{s} \frac{1}{m} \leqslant ||y_1 - x^*|| \right\}. \tag{3.17}$$

Proof. Set $\beta_n := k_n - 1$. From (3.11) and property of $\Pi_K T$, we have that

$$\begin{aligned} ||y_{n+1} - x^*|| &= ||(\Pi_K T_n)^n y_n - (\Pi_K T)^n x^*|| \\ &\leqslant ||(\Pi_K T)^n y_n - (\Pi_K T)^n x^*|| + ||(\Pi_K T_n)^n y_n - (\Pi_K T)^n y_n|| \\ &\leqslant k_n ||y_n - x^*|| - \psi(||y_n - x^*||) + ||(\Pi_K T_n)^n y_n - (\Pi_K T)^n y_n|| \\ &= (1+\beta_n)||y_n - x^*|| - \psi(||y_n - x^*||) + ||(\Pi_K T_n)^n y_n \\ &\quad - (\Pi_K T)^n y_n||. \end{aligned} \tag{3.18}$$

Now, suppose (1) is satisfied. Then we have $\{T(\Pi_K T)^{n-1} y_n\}$ is bounded, and hence by (3.12) $\{T_n(\Pi_K T_n)^{n-1} y_n\}$ is bounded.

Thus, by inequality (2.5)

$$\begin{aligned} &||J(T_n(\Pi_K T_n)^{n-1} y_n) - J(T(\Pi_K T)^{n-1} y_n)|| \\ &\quad \leqslant 8Rh_E(16L||T_n(\Pi_K T_n)^{n-1} y_n - T(\Pi_K T)^{n-1} y_n||/R) \\ &\quad \leqslant 8Rh_E\left(\frac{16L}{R}(h_n g(M) + \delta_n)\right). \end{aligned}$$

This implies, by property (c) of Π_K, that

$$\begin{aligned}
&\|(\Pi_K T_n)^n y_n - (\Pi_K T)^n y_n\| \\
&\leqslant 4LRg_E^{-1}(\|J(T_n(\Pi_K T_n)^{n-1} y_n) - J(T(\Pi_K T)^{n-1} y_n)\|/2R) \\
&\leqslant 4RLg_E^{-1}\left(4h_E\left(\frac{16L}{R}(h_n g(M) + \delta_n)\right)\right)
\end{aligned}$$

and hence from the property of g_E^{-1} and h_E we have that $\gamma_n := \|(\Pi_K T_n)^n y_n - (\Pi_K T)^n y_n\| \to 0$ as $h_n,\ \delta_n \to 0$.

Therefore, using inequality (3.18) all conditions of Lemma 2.4 are satisfied with $\alpha_i = 1\ \forall i \geqslant 1$. So, the conclusions hold with

$$\bar{\gamma}_n := \gamma_n + \beta_n M' \text{ for some } M' > 0. \tag{3.19}$$

Suppose (2) is satisfied. Then, setting $\lambda_n := \|y_n - x^*\|$, from (3.18) we get by Lemma 2.4 that the conclusions hold. □

Remark 3.7. Observe that if $\Pi_K T$ is weakly contractive then it is asymptotically weakly contractive with $k_n = 1$. Thus Theorem 3.6 extends Theorem 2.10 of [4]. Furthermore, the requirement that $0 \in K$ imposed in Theorem 2.10 of [4] is not needed in our more general Theorem 3.6.

3.2. Successive approximations in a Hilbert space

In a Hilbert space, the recursion formula (3.3) becomes

$$x_{n+1} = (P_K T)^n x_n, \quad n = 1, 2, \ldots, \ x_1 \in K. \tag{3.20}$$

We have the following theorem for asymptotically weakly contractive operators, whose proof follows as in the proof of Theorem 3.3.

Theorem 3.8. *Let K be a closed convex subset of the Hilbert space H, T be an asymptotically weakly contractive map from K to H of class $C_{\psi(t)}$ with $\{k_n\} \subseteq [1, \infty)$ such that $\sum(k_n - 1) < \infty$ and let $x^* \in F(T)$. Then, the assertion of Theorem* 3.3 *holds.*

Remark 3.9. For stability theorems for the perturbed approximations of (3.20) *in Hilbert spaces*, suppose that, instead of the exact operator T, we have some sequence $\{T_n\}$ of perturbed maps, $T_n : K \to H$ such that (3.12) is satisfied with P_K instead of Π_K. By considering the iteration process

$$y_{n+1} = (P_K T_n)^n y_n, \quad n = 1, 2, \ldots, \ y_1 \in K \tag{3.21}$$

and using the nonexpansive property of P_K one easily obtains a Hilbert space version of Theorem 3.6. In particular, Theorem 3.2 of [4] is a special case of this Hilbert space version in which $k_n = 1$ and $n = 1$ for all positive integers n. Furthermore, suppose $K_n \subseteq D(T), n \geqslant 1$ is a sequence of perturbed sets such that $\mathscr{H}(K_n, K) \leqslant \sigma_n$, where $\mathscr{H}$ is the Hausdorff metric and $\sigma_n \to 0$ as $n \to \infty$. By considering the iterative sequence $\{z_n\}$ defined by

$$z_{n+1} = P_{K_{n+1}} T(P_K T)^{n-1} z_n, \quad n = 1, 2, \ldots, \quad z_1 \in K \tag{3.22}$$

and using inequality (2.1), one obtains a generalization of Theorem 3.4 of [4] (where $k_n = 1$ and $n = 1$ for all positive integers n).

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References

[1] Ya. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Monotonic and Accretive Type, Marcel Dekker, New York, 1996, pp. 15–50.
[2] Ya. Alber, S. Guerre-Delabriere, Problems of fixed point theory in Hilbert and Banach spaces, Functional Differential Equations 2 (1994) 5–10.
[3] Ya. Alber, S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, Oper. Theory Adv. Appl. 98 (1997) 7–22.
[4] Ya. Alber, S. Guerre-Delabriere, On the projection methods for fixed point problems, Analysis 21 (2001) 17–39.
[5] Ya. Alber, S. Guerre-Delabriere, L. Zelenko, The principle of weakly contractive maps in metric spaces, Comm. Appl. Nonlinear Anal. 5 (1) (1998) 45–68.
[6] Ya. Alber, A. Notik, On some estimates for projection operator in Banach space, Comm. Appl. Nonlinear Anal. 2 (1) (1995) 47–56.
[7] Ya. Alber, S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, Panamerican Math. J. 4 (2) (1994) 39–54.
[8] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197–228.
[9] C.E. Chidume, An approximation method for monotone Lipschitzian operators in Hilbert spaces, J. Austral. Math. Soc. Ser. A 41 (1986) 59–63.
[10] C.E. Chidume, M.O. Osilike, Approximation methods for nonlinear operator equations of the m-accretive type, J. Math. Anal. Appl. 189 (1995) 225–239.
[11] T. Figiel, On the moduli of convexity and smoothness, Studia Math. 56 (1976) 121–155.
[12] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1) (1972) 171–174.
[13] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1) (1974) 147–150.
[14] J. Lindenstrauss, L. Tzafriry, Classical Banach Spaces II, Springer, Berlin, Heidelberg, New York, 1979.
[15] W.R. Mann, Mean value methods in iteration, Proc. Amer. Soc. 4 (1953) 506–510.
[16] Z.B. Xu, G.F. Roach, Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. 157 (1991) 189–210.